

# THE CLASSICAL STOCHASTIC IMPULSE CONTROL PROBLEM

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ABSTRACT. In this paper we study estimates for the solution to an obstacle problem arising in stochastic impulse control theory. We prove using different methods the known sharp  $C_{loc}^{1,1}$  estimate for the solution.

## 1. INTRODUCTION

Stochastic impulse control problems ([2], [14], [15], [9]) are control problems that fall between classical diffusion control and optimal stopping problems. In such problems the controller is allowed to instantaneously move the state process by a certain amount every time the state exits the non-intervention region. This allows for the controlled process to have sample paths with jumps. There is an enormous literature studying stochastic impulse control models and many of these models have found a wide range of applications in electrical engineering, mechanical engineering, quantum engineering, robotics, image processing, and mathematical finance. A key operator in stochastic impulse control problems is the intervention operator

$$(1) \quad Mu(x) = \inf_{\xi \geq 0} (u(x + \xi) + 1).$$

The operator represents the value of the strategy that consists of taking the best immediate action in state  $x$  and behaving optimally afterward. Since it is not always optimal to intervene, this leads to the quasi-variational inequality

$$(2) \quad u(x) \leq Mu(x) \quad \forall x \in \mathbb{R}^n.$$

From the analytic perspective one obtains an obstacle problem where the obstacle depends implicitly and nonlocally on the solution. More precisely we can consider the classical stochastic impulse control problem

$$(3) \quad \begin{cases} \Delta u(x) \geq f(x) & \forall x \in \Omega. \\ u(x) \leq Mu(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

Here we let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with a  $C^{2,\alpha}$  boundary  $\partial\Omega$ ,  $u \in W_0^{1,2}(\Omega)$ ,  $f \in L^\infty(\Omega)$ ,  $f \geq 0$ , and

$$(4) \quad Mu(x) = \inf_{\substack{\xi \geq 0 \\ x+\xi \in \bar{\Omega}}} (u(x + \xi) + 1).$$

The assumption  $f \geq 0$  implies that the solution  $\bar{u}$  to the boundary value problem  $\Delta \bar{u} = f$  in  $\Omega$  with  $\bar{u} \in H_0^1(\Omega)$  satisfies  $\bar{u} \geq 0$ . This implies in particular that the set of solutions to  $v \leq M\bar{u}$  with  $v \in H_0^1(\Omega)$  is nonempty. This allows for an iterative procedure to prove existence and uniqueness of the solution ([2]). We also point out that the sharp  $C_{loc}^{1,1}$  estimate in the classical stochastic impulse control problem has been previously obtained ([5], [6]). In this paper we present a new proof for the sharp  $C_{loc}^{1,1}(\Omega)$  estimate for the solution to the classical stochastic impulse control problem. As a corollary of our proof we also obtain a direct proof of the fact that the nonlocal obstacle,  $Mu(x)$  is  $C_{loc}^{1,1}$  on the contact set  $\{u = Mu\}$ . Since the obstacle depends on the solution, we have to treat the regularity question for both the solution and the obstacle. The strategy is to improve the regularity of one and use it to improve the regularity of the other. We start by proving continuity of the solution and then proceed to prove an almost concavity estimate for the obstacle. In the last section we use the almost concavity estimate and superharmonicity to produce the  $C^{1,1}$  estimate.

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## 2. PRELIMINARY ESTIMATES

We list some known properties of our obstacle  $Mu$  that will be useful in this section. We refer to ([1], [2]) for relevant details.

1.  $u_1 \leq u_2$  a.e.  $\Rightarrow Mu_1 \leq Mu_2$  a.e.
2.  $M(tu + (1-t)v) \geq tMu + (1-t)Mv$  for all  $t \in [0, 1]$ .
3.  $M : L^\infty \rightarrow L^\infty$ .
4.  $\|Mu - Mv\|_{C(\Omega)} \leq \|u - v\|_{C(\Omega)}$  for all  $u, v \in C(\Omega)$ .

We recall the definition of subsolutions. Let  $\psi$  is a sufficiently regular function,  $a(u, v) = \int_\Omega \nabla u \nabla v \, dx$  and  $(u, v) = \int_\Omega uv \, dx$ . The set of subsolutions is defined to be the set of functions  $z$  satisfying

1.  $z \in H_0^1$ , such that  $z \leq \psi$ .
2.  $a(z, \varphi) \geq (f, \varphi)$ ,  $\forall \varphi \in H_0^1$ ,  $\varphi \geq 0$ .

We also recall the comparison principle for variational inequalities.

**Lemma 1.** (*Comparison Principle*) *The unique solution to the variational inequality*

$$(5) \quad \begin{cases} a(u, v) \geq (f, u - v) & \forall x \in \Omega, \\ u \leq \psi & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega, \end{cases}$$

*is a subsolution and the maximum element among subsolutions.*

The estimate we are interested in proving in this section is the continuity of the solution for the classical stochastic impulse control problem. We first recall an estimate for solutions that proves that solutions to variational inequalities are

continuous across the free boundary. It is sufficient to assume that the obstacle,  $\psi$ , is continuous. We refer to ([3]) for a proof.

**Lemma 2.** (*Evans Lemma*) *Let  $u$  be a solution to (5) with continuous obstacle  $\psi$ . Then  $u \in C(\Omega)$ .*

We now proceed to prove the existence of a unique continuous solution to the classical stochastic impulse control problem. We follow closely the proof in ([13]).

**Theorem 1.** *There exists a unique solution  $u \in C(\Omega)$  of (3).*

*Proof.* From standard elliptic theory we know that there exists a unique solution  $u_0 \in C(\Omega)$  of

$$(6) \quad \begin{cases} a(u, v) = (f, u - v) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

Since  $Mu_0$  is continuous we know from Evans Lemma that there exists a unique solution  $u_1 \in C(\Omega)$  of

$$(7) \quad \begin{cases} a(u, v) \geq (f, u - v) & \forall x \in \Omega. \\ u \leq Mu_0 & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

Moreover for  $n = 2, 3, \dots$  we obtain  $u_n \in C(\Omega)$  satisfying

$$(8) \quad \begin{cases} a(u, v) \geq (f, u - v) & \forall x \in \Omega. \\ u \leq Mu_{n-1} & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

Since  $u_1$  is a subsolution of (6), by (Lemma 1), we know that  $u_1 \leq u_0$ . We also know that 0 is a subsolution of (7), hence (Lemma 1) implies that  $0 \leq u_1$ . Moreover it follows from the properties of  $Mu$  that  $0 \leq Mu_1 \leq Mu_0$ . This implies in particular that  $u_2$  is an admissible subsolution to (7). Arguing as before we see that  $0 \leq u_2 \leq u_1$ . We can continue this process and obtain a sequence of functions

$$(9) \quad 0 \leq \dots \leq u_n \leq \dots \leq u_1 \leq u_0.$$

Now we look to prove an upper bound on the sequence. Consider  $\mu \in (0, 1)$  such that  $\mu \|u_0\|_{C(\Omega)} \leq 1$ . Assume there exists  $\theta_n \in (0, 1]$  such that  $\forall n \in \mathbb{N}$ ,

$$(10) \quad u_n - u_{n+1} \leq \theta_n u_n.$$

We claim that this implies

$$(11) \quad u_{n+1} - u_{n+2} \leq \theta_n (1 - \mu) u_{n+1}.$$

With this claim we are able to almost conclude the proof of the theorem. In particular the positivity of  $u_n$  implies that  $u_1 - u_2 \leq u_2$ . We can set  $\theta_1 = 1$ . Moreover from (11) it follows that  $u_2 - u_3 \leq (1 - \mu)u_2$ . Hence  $\theta_2 = (1 - \mu)$ . Therefore setting  $\theta_n = (1 - \mu)^{n-1}$  we find

$$(12) \quad u_{n+1} - u_{n+2} \leq (1 - \mu)^n u_{n+1} \leq (1 - \mu)^n \|u_0\|_{C(\Omega)}.$$

Combining (12) with (9) we see that there exists a function  $u \in C(\Omega)$  such that  $\|u_n - u\|_{C(\Omega)} \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover from the estimate  $\|Mu - Mv\|_{C(\Omega)} \leq \|u - v\|_{C(\Omega)}$  it follows that  $u$  is a solution to the classical stochastic impulse control problem. Hence we are reduced to proving (11) and establishing uniqueness of the solution. By the concavity of  $Mu$  and (10) it follows,

$$(*) \quad \psi = (1 - \theta_n)Mu_n + \theta_n \leq (1 - \theta_n)Mu_n + \theta_n M0 \leq M(1 - \theta_n u_n) \leq Mu_{n+1}.$$

We consider the continuous solutions to the following obstacle problems. Let  $w \in C(\Omega)$  solve,

$$(13) \quad \begin{cases} a(u, v) \geq (f, u - v) & \forall x \in \Omega. \\ u \leq \psi & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

Let  $z \in C(\Omega)$  solve,

$$(14) \quad \begin{cases} a(u, v) \geq (f, u - v) & \forall x \in \Omega. \\ u \leq 1 & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

From (\*) and the comparison theorem for variational inequalities it follows that  $w \leq u_{n+2}$ . Moreover it follows that  $\theta_n z$  solves,

$$(15) \quad \begin{cases} a(u, v) \geq (f, u - v) & \forall x \in \Omega. \\ u \leq \theta_n & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

Observing that  $\psi \geq \theta_n$ , it follows from comparison that  $\theta_n w \geq \theta_n z$ . Next we observe that  $(1 - \theta_n)u_{n+1}$  is a subsolution and  $(1 - \theta_n)w$  is a solution of the following obstacle problem,

$$(16) \quad \begin{cases} a(u, v) \geq (f, u - v) & \forall x \in \Omega. \\ u \leq (1 - \theta_n)\psi & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases}$$

Hence we find,  $(1 - \theta_n)u_{n+1} \leq (1 - \theta_n)w$ . Putting this together we obtain,

$$(**) \quad (1 - \theta_n)u_{n+1} + \theta_n z \leq (1 - \theta_n)w + \theta_n w = w \leq u_{n+2}.$$

Recall that  $\forall n, \mu u_{n+1} \leq 1$ . This implies that  $\mu u_{n+1}$  is a subsolution of (14). So in particular,  $\mu u_{n+1} \leq z$ . Putting this into (\*\*) we obtain our desired estimate (11),

$$u_{n+1} - u_{n+2} \leq \theta_n(1 - \mu)u_{n+1}.$$

Finally to prove uniqueness, suppose  $u$  and  $\bar{u}$  are distinct solutions. The positivity of the solution implies  $u - \bar{u} \leq u$ . Hence arguing as above we find  $u - \bar{u} \leq (1 - \mu)^n u$ , for all  $n \geq 0$ . Letting  $n \rightarrow \infty$  we find that  $u - \bar{u} \leq 0$ . Interchanging  $u$  and  $\bar{u}$  we conclude  $u = \bar{u}$ .  $\square$

Using the improved regularity on the solution  $u$ , we now proceed to prove that the obstacle  $Mu(x)$  is semi-concave with semi-concavity modulus  $\omega(r) = Cr^2$ .

**Theorem 2.** *Let  $\varphi(x) \in C^{1,1}(\Omega)$ , strictly positive, bounded, and decreasing in the positive cone  $\xi \geq 0$ . Then the obstacle*

$$Mu(x) = \varphi(x) + \inf_{\substack{\xi \geq 0 \\ x + \xi \in \Omega}} u(x + \xi)$$

*is locally semi-concave with a semi-concavity modulus  $\omega(r) = Cr^2$ .*

*Proof.* We consider two distinct cases:

1.  $x_0 \in \{u = Mu\}$ .
2.  $x_0 \in \{u < Mu\}$ .

**Case 1:** Fix  $x_0 \in \{u = Mu\}$ .

The proof in this case is based on characterizing the set where the infimum of  $u$  occurs and establishing that this set is uniformly contained in the non-contact region  $\{u < Mu\}$ . This is the content of the following claims. We define the following sets:

1.  $\Sigma_{\geq x_0} = \{x_0 + \xi : \xi \geq 0\}$ .
2.  $\Sigma_{x_0} = \{\varphi(x_0) + u(x_0 + \xi) = Mu(x_0)\}$ .

The following claim characterizes  $\Sigma_{x_0}$  as the set of points where  $u$  realizes its infimum.

**Claim 1.** *For every  $y \in (\Sigma_{\geq x_0} \setminus \Sigma_{x_0})$  and for every  $x \in \Sigma_{x_0}$ ,  $u(x) \leq u(y)$ .*

*Proof.* Fix  $\bar{x} \in \Sigma_{x_0}$ . Suppose by contradiction that  $\exists x_1 \in \Sigma_{\geq x_0} \setminus \Sigma_{x_0}$  such that  $u(x_1) < u(\bar{x})$ . This implies the following chain of inequalities,

$$\begin{aligned} \varphi(x_0) + u(x_1) &< \varphi(x_0) + u(\bar{x}) \\ &= Mu(x_0) = \varphi(x_0) + \inf_{\substack{\xi \geq 0 \\ x_0 + \xi \in \Omega}} u(x_0 + \xi). \end{aligned}$$

In particular we obtain,

$$u(x_1) < \inf_{\substack{\xi \geq 0 \\ x_0 + \xi \in \Omega}} u(x_0 + \xi).$$

This is our desired contradiction.  $\square$

We now prove that pointwise the elements of  $\Sigma_{x_0}$  are contained in the non-contact region,  $\{u < Mu\}$ .

**Claim 2.** *Suppose the solution to the Boundary Value Problem  $\Delta \bar{u} = 0$  satisfies*

$$\bar{u} < \inf_{\partial\Omega} \varphi.$$

*Then  $\forall x \in \Sigma_{x_0}$  it follows that  $u(x) < Mu(x)$ .*

*Moreover in a neighborhood  $N_1$  of  $x$  we have  $u \in C^{1,1}(N_1)$*

*Proof.* We observe that the first statement ensures that  $\Sigma_{x_0} \cap (\partial\Omega) = \emptyset$ . Suppose  $x_0 \in \Omega^\circ$ ,  $x \in \partial\Omega$  and  $x_0 \leq x$ . Then we observe,

$$\begin{aligned} Mu(x_0) &= u(x_0) \\ &\leq \bar{u}(x_0) < \inf_{\partial\Omega} \varphi \leq \varphi(x) + u(x) \leq \varphi(x_0) + u(x). \end{aligned}$$

The last inequality follows because  $\varphi(x)$  is monotonically decreasing in the cone. Hence in particular  $\Sigma_{x_0} \cap (\partial\Omega) = \emptyset$ .

Suppose now by contradiction that  $\exists x \in \Sigma_{x_0}$  such that  $u(x) = Mu(x)$ . Then we have the following chain of inequalities,

$$\begin{aligned} u(x_0) &= Mu(x_0) \\ &= \varphi(x_0) + u(x) \\ &= \varphi(x_0) + Mu(x) \geq \varphi(x_0) + Mu(x_0) > Mu(x_0) \end{aligned}$$

The last inequality follows from the strict positivity of the function  $\varphi$ . But we observe that the inequality contradicts the obstacle constraint  $u(x_0) \leq Mu(x_0)$ . Hence we have reached our desired contradiction.

Finally the last statement of the claim follows from the continuity of  $u$ . The continuity of the solution implies that  $\{u < Mu\}$  is an open set and thus in a small neighborhood  $N_1$  of  $x$ ,  $u$  satisfies the equation,  $\Delta u = 0$ . We can therefore apply interior regularity estimates to conclude.  $\square$

We now strengthen the previous claim to obtain a uniform neighborhood of  $\Sigma_{x_0}$  that is strictly contained in the non-contact region.

**Claim 3.**  *$\exists \delta_0 > 0$  such that  $d\{\{u = Mu\}, \Sigma_{x_0}\} > \delta_0$ .*

*Proof.* Suppose by contradiction  $\exists \{\delta_k\} \searrow 0$  and  $\{x_k\} \subset \Sigma_{x_0}$ , such that

$$d(x_k, \{u = Mu\}) < \delta_k.$$

By definition,  $x_k \in \Sigma_{x_0}$ , implies

$$\varphi(x_0) + u(x_k) = Mu(x_0) \quad \forall k.$$

By the continuity of  $u(x)$  this implies in particular that  $\varphi(x_0) + u(\bar{x}) = Mu(x_0)$  for some  $\bar{x} \in \{u = Mu\}$ . On the other hand,  $\varphi(x_0) + u(\bar{x}) = Mu(x_0)$  implies  $\bar{x} \in \Sigma_{x_0}$ . Hence from the previous claim we obtain,

$$Mu(\bar{x}) = u(\bar{x}) < Mu(\bar{x}).$$

This is our desired contradiction.  $\square$

We now state and prove a claim which allows us to redefine the obstacle in the neighborhood of a contact point.

**Claim 4.** *For every  $x, \bar{x} \in \Omega$ ,  $\exists \delta > 0$ , such that if  $|x - x_0| < \delta$ , and  $d(\bar{x}, \Sigma_{x_0}) > \delta$ , then  $u(x) < \varphi(x) + u(\bar{x})$ . Moreover, if  $x \in \{u = Mu\}$ , then  $\bar{x} \notin \Sigma_x$ .*

*Proof.* Suppose by contradiction that there exists a sequence of points  $\{x_k\}$  and  $\{\bar{x}_{k'}\}$  satisfying:

1.  $|x_k - x_0| = \delta_k$ .
2.  $d(\bar{x}_{k'}, \Sigma_0) > \delta_{k'} > 0$ .
3.  $\{\delta_k\} \searrow 0$  and  $\{\delta_{k'}\} \searrow 0$ .
4.  $u(x_k) \geq \varphi(x_k) + u(\bar{x}_{k'}) \quad \forall k$  and  $\forall k'$ .

We observe that from the previous claim  $\exists k_0, k'_0$ , such that  $\forall k \geq k_0$  we have the following chain of inequalities,

$$\begin{aligned} Mu(x_0 + \delta_k) &\leq Mu(\bar{x}_{k'_0}) \\ &\leq \varphi(\bar{x}_{k'_0}) + u(\bar{x}_{k'_0}) \\ &\leq \varphi(x_0 + \delta_k) + u(\bar{x}_{k'_0}) \\ &\leq u(x_0 + \delta_k) \leq Mu(x_0 + \delta_k). \end{aligned}$$

Thus the above inequalities are all equalities. This implies  $\forall k \geq k_0$ ,

$$Mu(x_0 + \delta_k) = \varphi(x_0 + \delta_k) + u(\bar{x}_{k'_0}).$$

Letting  $k \rightarrow \infty$  we obtain,

$$Mu(x_0) = \varphi(x_0) + u(\bar{x}_{k'_0}).$$

Which implies in particular that  $\bar{x}_{k'_0} \in \Sigma_{x_0}$ . This is our desired contradiction.  $\square$

From the last claim we can redefine the obstacle for  $V_\delta = \{|x - x_0| < \delta\}$ . In particular by taking  $\delta$  sufficiently small  $\exists N_2$  neighborhood of  $\Sigma_{x_0}$  such that,

$$Mu(x) = \varphi(x) + \inf_{\substack{\xi \geq 0 \\ x + \xi \in N_2}} u(x + \xi).$$

For an even smaller  $\delta$ ,

$$Mu(x) = \varphi(x) + \inf_{\substack{\xi \geq 0 \\ x_0 + \xi \in N_3}} u(x + \xi).$$

Where  $N_3$  is such that,

$$V_\delta + N_3 - x_0 \subseteq N_1.$$

Here  $N_1$  is the neighborhood obtained in Claim 3. In particular for  $x \in V_\delta$  and  $\xi \in N_3 - x_0$ ,

$$\frac{\partial^2}{\partial \eta^2} (u(x + \xi) - C|x|^2) \leq 0.$$

Now the infimum of concave functions is concave, so we have in particular,

$$\frac{\partial^2}{\partial \eta^2} \inf_{\substack{\xi \geq 0 \\ \xi \in N_3 - x_0}} (u(x + \xi) - C|x|^2) \leq 0, \quad \forall x \in V_\delta.$$

This implies,

$$\begin{aligned} \frac{\partial^2}{\partial \eta^2} Mu(x) &= \frac{\partial^2}{\partial \eta^2} \varphi(x) + \frac{\partial^2}{\partial \eta^2} \inf_{\substack{\xi \geq 0 \\ \xi \in N_3 - x_0}} u(x + \xi) \\ &\leq \left\| \frac{\partial^2}{\partial \eta^2} \varphi \right\|_\infty + C \leq C. \end{aligned}$$

Hence, in a neighborhood of a contact point our obstacle is semi-concave.

**Case 2:** Fix  $x \in \{u < Mu\}$ . We observe that the infimum of  $u$  in the positive cone,  $\xi \geq 0$ , must always be realized at a non-contact point. Suppose  $\exists x + \xi_1 \in \{u = Mu\}$  satisfying,

$$\inf_{\substack{\xi \geq 0 \\ x + \xi \in \Omega}} u(x + \xi) = u(x + \xi_1).$$

Then from **Case 1** there exists  $\xi_2 \in \Sigma_{x+\xi_1} \subset \{u < Mu\}$  such that,

$$\inf_{\substack{\xi \geq 0 \\ x + \xi_1 + \xi \in \Omega}} u(x + \xi_1 + \xi) = u(x + \xi_1 + \xi_2).$$

Since  $\xi_1 + \xi_2 \geq 0$ , we have found a positive vector admissible to

$$\inf_{\substack{\xi \geq 0 \\ x + \xi \in \Omega}} u(x + \xi).$$

Furthermore,  $u(x + \xi_1 + \xi_2) \leq u(x + \xi_1)$ . Hence we conclude that for a fixed  $x \in \{u < Mu\}$  and for some  $x + \bar{\xi}$  in  $\{u < Mu\}$ ,

$$\inf_{\substack{\xi \geq 0 \\ x + \xi \in \Omega}} u(x + \xi) = u(x + \bar{\xi}).$$

Moreover from Claim 3 we know that  $x + \bar{\xi}$  is a uniform positive distance away from the contact set  $\{u = Mu\}$ . Hence there exists a uniform neighborhood  $N_0$  of points around  $x + \bar{\xi}$  where  $\{u < Mu\}$ . In a smaller neighborhood  $N_1$ ,  $u \in C^{1,1}(N_1)$ . As before, for  $x + \xi \in N_1$ ,

$$\frac{\partial^2}{\partial \eta^2} (u(x + \xi) - C|x|^2) \leq 0.$$

This implies,

$$\begin{aligned} \frac{\partial^2}{\partial \eta^2} Mu(x) &= \frac{\partial^2}{\partial \eta^2} \varphi(x) + \frac{\partial^2}{\partial \eta^2} \inf_{\substack{\xi \geq 0 \\ x + \xi \in \Omega}} u(x + \xi) \\ &\leq \left\| \frac{\partial^2}{\partial \eta^2} \varphi \right\|_\infty + C \leq C. \end{aligned}$$

Hence, in a neighborhood of a non-contact point our obstacle is semi-concave.  $\square$

### 3. OPTIMAL REGULARITY FOR THE STOCHASTIC IMPULSE CONTROL PROBLEM

In the previous section we proved that the unique bounded solution to the classical stochastic impulse control problem is continuous and that our implicit constraint obstacle is locally semi-concave. We now consider the sharp  $C^{1,1}$  estimate for the solution.



**Theorem 3.** *Let  $u$  be the unique continuous solution of the classical stochastic impulse control problem. Then  $u \in C_{loc}^{1,1}(\Omega)$ .*

We set  $Mu(x) = \varphi_u(x)$ . Recall that a function  $v$  is semi-concave with semi-concavity modulus  $\omega(r)$  if a vector  $p \in \mathbb{R}^n$  belongs to  $D^+v(x)$  if and only if  $v(y) - v(x) - \langle p, y - x \rangle \leq \omega(|x - y|)$ . Fix  $x_0 \in \{u = \varphi_u\}$ . Define the linear part of the obstacle,  $L_{x_0}(x) = \varphi_u(x_0) + \langle p, x - x_0 \rangle$ . We consider

$$w(x) = u(x) - L_{x_0}(x).$$

We observe that in  $B_r(x)$ ,  $w(x)$  has a modulus of semi-concavity  $\omega(r) = Cr^2$ , i.e.  $w(x) \leq Cr^2$ . We now state our main lemma.

**Lemma 3.** *There exists universal constants  $K, C > 0$ , such that  $\forall x \in B_{r/4}(x_0)$ ,*

$$(17) \quad -K \leq \Delta w \leq C.$$

Before proving this lemma we make a few observations. Fix  $\Phi \in C_0^\infty(B_{\frac{r}{2}}(x_0))$ . We recall the following fact from the theory of distributions: If  $u$  is a negative distribution in  $X$  with  $u(\Phi) \leq 0$  for all non-negative  $\Phi \in C_0^\infty(X)$ , then  $u$  is a negative measure. In particular we have,

$$(18) \quad 0 \geq \int_{B_{\frac{r}{2}}} \Phi d\mu = \int_{B_{\frac{r}{2}}} \Delta u \Phi.$$

We consider  $\forall \rho < \frac{r}{2}$ ,

$$(19) \quad \frac{\mu(B_\rho(x_0))}{|B_\rho(x_0)|} = \frac{1}{\alpha(n)\rho^n} \int_{B_\rho} d\mu = \frac{1}{\alpha(n)\rho^n} \int_{B_\rho} \Delta u.$$

A straightforward application of the Gauss-Green Formula gives to us the following identity,

$$(20) \quad \frac{1}{\alpha(n)\rho^n} \int_{B_\rho} \Delta w = \frac{n}{\rho} \frac{d}{d\rho} \Psi(\rho).$$

Where  $\Psi(\rho) = \frac{1}{n\alpha(n)\rho^{n-1}} \int_{\partial B_\rho} w$ . Before proving the main lemma we will first prove the following claim.

**Claim 5.** *Let  $w = u - L_{x_0}$  be defined as before. Then for some universal constant  $K(n) > 0$ ,*

$$\frac{n}{\rho} \frac{d}{d\rho} \Psi(\rho) \geq -K.$$

*Proof.* We expand the derivative and compute.

$$\begin{aligned}
\frac{n}{\rho} \frac{d}{d\rho} \Psi(\rho) &= \frac{n}{\rho} \frac{1-n}{n\alpha(n)\rho^n} \int_{\partial B_\rho(x_0)} w(y) dS(y) + \frac{n}{n\alpha(n)\rho^n} \frac{d}{d\rho} \int_{\partial B_\rho(x_0)} w(y) dS(y) \\
&= \frac{n}{\rho} \frac{n-1}{n\alpha(n)\rho^n} \int_{\partial B_\rho(x_0)} -w(y) dS(y) + \frac{1}{\alpha(n)\rho^n} \frac{d}{d\rho} \rho^{n-1} \int_{\partial B_1(0)} w(x_0 + \rho z) dS(z) \\
&= \frac{n}{\rho} \frac{n-1}{n\alpha(n)\rho^n} \int_{\partial B_\rho(x_0)} -w(y) dS(y) + \frac{\rho^{n-2}(n-1)}{\alpha(n)\rho^n} \frac{\rho^{n-1}}{\rho^{n-1}} \int_{\partial B_1(0)} w(x_0 + \rho z) dS(z) \\
&\quad + \frac{\rho^{n-1}}{\alpha(n)\rho^n} \frac{d}{d\rho} \int_{\partial B_1(0)} w(x_0 + \rho z) dS(z) \\
&= \frac{n}{\rho} \frac{n-1}{n\alpha(n)\rho^n} \int_{\partial B_\rho(x_0)} -w(y) dS(y) + \frac{(n-1)}{\alpha(n)\rho^{n+1}} \int_{\partial B_\rho(x_0)} w(y) dS(y) \\
&\quad + \frac{\rho^{n-1}}{\alpha(n)\rho^n} \frac{d}{d\rho} \int_{\partial B_1(0)} w(x_0 + \rho z) dS(z)
\end{aligned}$$

Now we proceed to estimate each integral. By the modulus of semi-concavity on the ball we have,

$$\frac{n}{\rho} \frac{n-1}{n\alpha(n)\rho^n} \int_{\partial B_\rho(x_0)} -w(y) dS(y) \geq \frac{n(n-1)}{\alpha(n)n\rho^{n+1}} |\partial B_\rho(x_0)| (-C\rho^2) = -C(n^2 - n).$$

By the mean value theorem for subharmonic functions we have,

$$\frac{(n-1)}{\alpha(n)\rho^{n+1}} \int_{\partial B_\rho(x_0)} w(y) dS(y) \geq \frac{(n-1)}{\alpha(n)\rho^{n+1}} w(x_0) = 0.$$

By the nondecreasing property for the average integral we have:

$$\frac{\rho^{n-1}}{\alpha(n)\rho^n} \frac{d}{d\rho} \int_{\partial B_1(0)} w(x_0 + \rho z) dS(z) \geq 0.$$

Hence for  $K = C(n^2 - n)$  we obtain the desired estimate.  $\square$

*Proof.* (Lemma 3) From the claim we obtain the estimate,

$$\frac{1}{\alpha(n)\rho^n} \int_{B_\rho} \Delta w \geq -K.$$

Moreover from (19) and the semi-concavity estimate from above we know,

$$C \geq \frac{\mu(B_\rho(x_0))}{|B_\rho(x_0)|} \geq -K.$$

Letting  $\rho \rightarrow 0$  we find  $\forall x \in B_{\frac{r}{4}}(x_0)$ ,

$$C \geq \Delta u(x) \geq -K.$$

$\square$

We now state and prove the sharp estimate for the solution.

**Theorem 4.** *Let  $u$  be a solution to the classical stochastic impulse control problem. Then,*

$$(21) \quad \|u\|_{C^{1,1}(B_{r/4})} \leq C$$

*Proof.* We recall some basic notions and definitions for convenience. For further details refer to ([4]). We say that  $P$  is a paraboloid of opening  $M$  whenever,

$$P(x) = l_0 + l(x) \pm \frac{M}{2}|x|^2.$$

We define,

$$\overline{\Theta}(u, A)(x_0),$$

to be the infimum of all positive constants  $M$  for which there is a conex paraboloid of opening  $M$  that touches  $u$  from above at  $x_0$  in  $A$ . Similarly one can define the infimum of all positive constants  $M$  for which there is a convex paraboloid of opening  $-M$  that touches  $u$  from below at  $x_0$  in  $A$ ,

$$\underline{\Theta}(u, A)(x_0).$$

We further define,

$$\Theta(u, A)(x_0) = \sup\{\overline{\Theta}(u, A)(x_0), \underline{\Theta}(u, A)(x_0)\} \leq \infty.$$

As before we fix  $x_0 \in \{u = Mu\}$ . We consider the second incremental quotients of  $u$  and  $Mu$ ,

$$\begin{aligned} \Delta_h^2 u(x_0) &= \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{|h|^2}, \\ \Delta_h^2 Mu(x_0) &= \frac{Mu(x_0 + h) + Mu(x_0 - h) - 2Mu(x_0)}{|h|^2}. \end{aligned}$$

We make the following observations,

1.  $\Delta_h^2 u(x_0) \leq \Delta_h^2 Mu(x_0)$ .
2.  $0 \leq \overline{\Theta}(u, B_\rho)(x_0) = \overline{\Theta}(Mu, B_\rho)(x_0) \leq C$ .
3.  $0 \leq \underline{\Theta}(u, B_\rho)(x_0) = \underline{\Theta}(Mu, B_\rho)(x_0) \leq K$ .

Putting the estimates together we obtain,

$$-K \leq -\underline{\Theta}(u, B_\rho)(x_0) \leq \Delta_h^2 u(x_0) \leq \Delta_h^2 Mu(x_0) \leq \overline{\Theta}(Mu, B_\rho)(x_0) \leq C.$$

In particular  $\forall x \in B_\rho$ ,

$$-K \leq -\underline{\Theta}(u, B_\rho)(x) \leq \Delta_h^2 u(x) \leq \overline{\Theta}(u, B_\rho)(x) \leq C.$$

This follows from choosing  $\forall x \in B_\rho$ , the lower paraboloid and upper paraboloid to be respectively,

$$\begin{aligned} P_1(y) &= u(x) + \langle p_1, y - x \rangle - \frac{K}{2}|y|^2. \\ P_2(y) &= u(x) + \langle p_2, y - x \rangle + \frac{C}{2}|y|^2. \end{aligned}$$

Hence we obtain,

$$\Theta(u, \epsilon) = \Theta(u, B_\rho \cap B_\epsilon(x))(x) \in L^\infty(B_\rho).$$

This implies,

$$\|D^2 u\|_{L^\infty(B_\rho)} \leq C.$$

In particular we obtain our desired estimate,

$$\|u\|_{C^{1,1}(B_\rho)} \leq C.$$

□

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